**MTH101   
Mathematics I (International)**

**Module 1  
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**Functions**

**Definition 1**

A **function** is a rule that takes certain numbers as inputs and assigns to exactly one output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

**Note**: A function can be considered as a set of ordered pairs .

**Notations**:

Let  be a function from *A* to *B* ()

*  represents domain of function 
*  represents range of function 
* Image of  is  since 

 is called a function from  **onto**  if 

Normally, we may present a function via four common ways:

1. Description (words)
2. Numeric (tables)
3. Visual (graphs)
4. Algebra (formulas)

**Example 1**

Consider a set . Is it a function?

Domain:

Range:

**Example 2**

Let  and .

So  and . We usually write.

The values of  at some points are as follow.













and 

**Example 3**

Let . Is  a function?

**Example 4** Find thedomain of the following functions.

(1) 

(2) 

(3) 

(4) 

**Example 5**

1) **** has the set of all real numbers as its domain and the interval  as its range.

2)  has the set of all real numbers as a domain and the interval as its range.

**Example 6**

**** or ****





**Definition 2** The function equals to the functionif and only if

1. 

2.  for all .

**Example 7** Check if the following functions are equal.

1) Let  and 

2) Let  and 

**Definition 3**

Let  and  be functions and .

**A composite function** of  and  (denoted by ) is a function  whose domain isand.

**Example 8** Let  and 

a) Let  Find  and domain of 

b) Let  Find  and domain of 

c) Let  Find  and domain of 

**Solutions**

a) The domain of  is  and domain of  is .

To find the domain of , we consider only  where  is in domain of . That is, .

Thus domain of is a set of  where  i.e. .

Then, the function  can be found by



**Symmetry**

**Definition 4** Let  be a function.

1. If ,  is called an **odd function** whose graph is symmetric about the origin.
2. If ,  is called an **even function** whose graph is symmetric about the *y*-axis.

**Example 9**

a) Let.

Consider .

Thus  is an odd function and it graph is shown in figure 1 below.



Figure 1

b) Let .

Consider .

Thus  is an even function whose graph shown in Figure 2.

  
Figure 2

**Inverse function**

**Definition 5** The function  is called a **one-to-one** function if and only if for all  if  and  then .

**Definition 6** Let  be a one-to-one function from  onto .

An inverse function of  is defined by  which is also a one-to-one function from  to .

**Remark** Graphs of  and  are symmetric about the line  as shown in Figure 3 below.

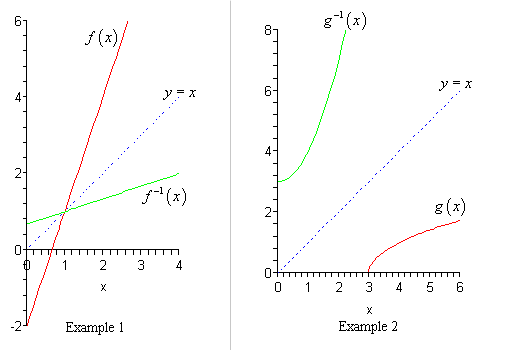


Figure 3

**Example10** Find an inverse of  where .

**Solution** From  (i.e. ), we have that  or 

We normally write  so that we can easily draw graphs of both functions  and  as follows









Figure 4

**Other Interesting Functions**

All functions here will be useful in the next sections.

**Algebraic Function**

**a. Polynomial Functions** are functions of the form



where  is a real number for each  and

 is a non-negative integer.

If  is the largest number such that , we call  a polynomial function of degree  such as  is a polynomial function of degree 3.

Normally, if there is nothing specific, the domain of a polynomial function is the set of all real numbers.

**b. Rational Functions** are functions formed by a ratio between two polynomial functions.

,

Note that, if there is nothing specific, the domain of this rational function is 

**Example 11** Let 

Rewrite function :  where 

Thus graph of  is the graph of, but undefined at 



Figure 5

**c. Functions of the form ** ; **** where the function **** is either a polynomial or a rational function.

The domain of this type of functions can be considered as follows

Case1 ****  is odd

The domain of **** is exactly the domain **** of ****

Case2 ****  is even

The domain of **** is  ****

**d. Functions formed by summation, multiplication and division of functions in part a. to c.**

Below are someexamples of functions in part c. and d.

1)  2) 

3) 

**Transcendental Functions**

**a. Exponential Functions** are functions of the form

**** where **** and ****

When **,**  its graph can be shown in Figure 6 below.



Figure 6

When ****, its graph can be shown in figure 7 below



Figure 7

**b. Logarithmic Function**

Logarithmic function is an inverse of exponential function. Given an exponential function ****. Then its inverse function is  or we can rewrite it as .

If   , then its graph is shown in Figure 8.

If   , then its graph is shown in Figure 9.



Figure 8



Figure 9

**Some facts about logarithmic functions**

1. Domain of a logarithmic function is  and its range is 

2. A logarithmic function is a one-to-one function.

3. 

4. Graph of  is a reflection of the graph **** across the line.

**Remark:** When  (where = natural number) **** has the inverse  which is normally written as  and it is called a natural logarithm.

The properties of **** and  are the same as of the followingproperties of**** and  (), respectively

**Properties of logarithmic and exponential functions**

Given positive numbers  where  and 

1. 

2. 

3.  and 

4. 

5. 

6. If , then 



7. 

8. 

9. 

10.  and  

11.  and  

**Example 12**  Find the values of 

(a)  (b) 

**c. Trigonometric Function**

 Graph of 

 Graph of 

 Graph of

Graph of 

Graph of 

 Graph of 

Normally, the inverse of a trigonometric function is not a function since each trigonometric function is not one-to-one. However, if we restrict the domain, we can make a one-to-one trigonometric function and define an inverse function as follows.

1) Restrict the domain of  to 

Its inverse function is.    

2) Restrict domain of  to 

Its inverse function is .

3) Restrict domain of  to 

Its inverse function is .

4) Restrict domain of  to 

Its inverse function is .

5) Restrict domain of  to 

Its inverse function is .

6) Restrict domain of  to 

Its inverse function is .

**Exercises on Functions**

1. Determine if the following are functions. Locate domain and range.

(a) 

(b) 

(c) 

(d) Let

|  |  |
| --- | --- |
| *x* | *y* |
| 15 | 2 |
| 2 | 13 |
| 13 | 13 |
| 5 | 3 |

1. Determine if each following function is either even or odd or neither.

(a) 

(b) 

(c) 

(d) 

1. What is the difference of  and ? Show in terms of composite functions.

**Answers to Function Exercises**

1. (a) yes  and 

(b) no  all real numbers

(c) yes  and 

(d) yes  and 

2. (a) odd (b) even

(c) odd (d) neither

3. Let  and 

  while

.

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**Limit and Continuity of Function**

**2.1 Limit of function**

Let  be a function. The limit of  when  approaches to  is not the value of  but it is a value that  is approaching to (as  approaches to ). There are two types of the limit.

**2.1.1 Limit of function as  (** is a **real number.)**

Suppose that  and  defined by the largest integer which is less than or equal to. For example,

 , ****, ****.

For some values of  which approaches to , the value  and are shown in Table 1.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | 0.5 | 0.9 | 0.99 | 0.999 | ... | 1.001 | 1.01 | 1.1 |
|  | 1.5 | 3.5 | 3.95 | 3.995 | ... | 4.005 | 4.05 | 4.5 |
|  | 0 | 0 | 0 | 0 | ... | 1 | 1 | 1 |

Table 1

We can see that when  approaches to ,  gets closer and closer to the value . However,  when  and  when . Thus  does not approach to one number.

Therefore, we say that  has the limit equal to 4 as  approaches to 1 and  does not have a limit when  approaches to 1. We may write them as

 and  does not exist.

The graph of the function  shows that the value of  gets closer to 4 when  approaches to 1. But the graph of the function  jumps from  to  at . Thus  has no limit at .

Using this concept, one can define the limit as follows:

**Definition** If  gets closer to  when  approaches to , we say that  is the limit of  when  approaches to , denoted by .

The values of  approaches to **** from two sides:

*  **approaches to  from the right side** isdenoted by

. In this case, we focus on when .

*  **approaches to from the left** **side** is denoted by

. In this case, we focus on  when .

From the above example, we have but and .

We see that the function  has the same limit from both sides when  approaches to 1 and

(Right limit) (Left limit) .

The following theorem guarantees the above remark.

**Theorem** 1  exists and equals to  if

(1) both and  exist and

(2) 

**Example 1** Compare  and .

**Solution**

**Properties of limits**

Let  and  be real numbers. Suppose that and. Then,

1. ,

2. ,

3. ,

4. ,

5. If  is a polynomial function, then for any number 

,

6.  where  is a natural number.

**Example 2** Evaluate **.**

**Solution**

**Example 3** Let be a function defined by 

Find the limits of  when  approaches 0 and 1.

**Solution**

**Example 4** Evaluate 

**Solution**

Sometimes, we find the limit by replacing  by  and may get the result in the form of . So, we can use these two techniques to find the limit.

1) Factoring

2) Conjugating

**Example 5** Calculate 

**Solution**

**Example 6** Calculate .

**Solution**

The following theorem is one of an important theorem that helps us to find the limit. It is typically used to confirm the limit of a function via comparison with two other functions whose limits are known or easily computed.

**Squeeze Theorem**

If  for all values of  at some points  and , then .



**Example 7** UsetheSqueeze Theoremto show that



**Example** 8

1. If  for , evaluate .
2. Calculate .

**Solution**

**Theorem**

1.  2. 

**Example 9**  Use  to show that 

**Proof** 





**Example 10**  Evaluate 

**Solution**

**2.1.2 Limit of function as  (infinity)**

When the domain of a function  is unbounded, the values of  may get closer to one value when  increases unboundedly (written as ****) or  decreases unboundedly (written as ****).

Let . Its graph can be shown here.



Consider the value of  in the following table.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | 100 | 1000 | 10000 | Increases unboundedly |
|  | 0.01 | 0.001 | 0.0001 |  |
|  | -100 | -1000 | -10000 | Decreases unboundedly |
|  | -0.01 | -0.001 | -0.0001 |  |

Table 2

We see that, when ****, the values of  get closer to 0 and  So, we say that limit of equals 0 as **** denoted by. Also, when ****, the values of  get closer to 0 as well, but . We say that limit of  equals 0 as **** and denote it by.

The above graph shows that  gets closer to -axis as  increases to infinity and decreases to negative infinity, but it never hit the -axis. We call a line that the graph gets closer to as an **asymptote** of the function.

**Properties of infinite limits**

Many properties of infinite limits are the same as those of limits at a finite number .

Let  and  be real numbers. Suppose that and Then,

1. 

2. ,

3. ,

4. 

5.  where  is positive and 

6.  where  is a positive integer.

**All 6 properties are the same when we replace  by **

**Example 1** Calculate

a) b) c) 

**Solution**

**Example 2** Evaluate 

**Solution**

**Example 3** Evaluate 

**Solution**

**Example 4** Calculate 

**Solution**

**Example 5** Calculate 

**Solution**

**Limit of a function associating with the number **

For any constant **,**

 and 

**Example 6** Calculate .

**Solution**

**2.2 Continuity of Function**

**Definition** A function  is continuous at  if all of the three following conditions are satisfied:

1.  exists,

2.  exists, (That is,.)

3. .

**Remark:** If at least one of the above conditions is not satisfied, then the given function is discontinuous at .

**Example 1** Let 

Consider the continuity of this function at :

1.  exists,

2.  exists, and

3. .

Thus,  is continuous at . Its graph is here.



**Example 2** Let  be a function defined by



Determine if this function is continuous at .

**Solution**

**Example 3** Let  be a function defined by



Find  that makes this function continuous at .

**Solution**

**Three Types of Discontinuities**

Consider the continuity of  at .

**1. Removable discontinuity**

It occurs when

(i) exists, but not equal to or

(ii) is undefined.

For example,  has a removable discontinuity at  as show in the Figure below.

***f(x) = x***

***f(0) = 1***

**2. Jump discontinuity or Ordinary discontinuity**

It occurs when does not exist due to the **unequal** existence of  and. For example, the function

 has a jump discontinuity at .

**1**

**-1**

**2**

**1**

**3. Infinite discontinuity**

It occurs when at least one of the left limit or the right limit does not exist. For example,  has an infinite discontinuity at  as shown here.



**Algebraic properties of functions on the continuity**

1. If  and  are continuous at , then  () and  (*k* is a constant) are also continuous at 

2. If  is continuous at  and , then



3. If  is continuous at  and is continuous at , then the composite function  is continuous at .

**Example 4** Let  be a function defined by



Locate where this function is continuous.

**Definition** If the function  is continuous everywhere in the interval , we say that  is continuous on .

**Definition** A function  is continuous in  where  if

1.  is continuous on ,

2.  and

3..

**Example 5** Let  be a function defined by 

Locate where this function is continuous.

**Solution**

**Limit and Continuity Exercises**

1. Find the limits of the following functions.

(a) Let. Find .

(b) 

(c) Let . Calculate .

(d) 

(e) 

2. Make the following functions continuous at .

(a)  , 

(b) 

3. Locate domain that makes the following function continuous.

(a) 

(b) 

4. Find  that makes  continuous everywhere.

5. Find  that makes each following limit exists.

(a) 

(b) 

(c) 

6. Compute the following limits.

(a) 

(b) 

(c) 

7. Compute the following limits.

(a) 

(b) 

(c) 

**Answers to limit and continuity exercises**

1. (a) 

(b) Does not exist

(c) –2

(d) 3

(e) –1 / 2

2. (a) add 

(b) add 

3. (a) 

(b) 

4. 1

5. (a) 5

(b) greater than or equal to 4

(c) less than or equal to 2

6. (a) 6

(b) –1

(c) –1/16

7. (a) 0

(b) 3/5

(c) 